A SIMPLIFIED SYSTEM OF SENTENTIAL LOGIC

Michael F. Goodman
Department of Philosophy
Humboldt State University, California, USA
mfg1@humboldt.edu

Abstract. *In thinking about reducing the symbolization of sentential logic to a minimum (say, to the tilde and the vel), where the conditional no longer appears in system, and hence in no proof in the system, one might wonder what would become of the Rule of Conditional Proof. Where this rule is said to be essential for the completeness of the system, some rule or other needs to be developed to do the work of Conditional Proof. The author attempts to accomplish this while specifying a minimum set of rules and axioms for {~, v}.*

Keywords: the Rule of Conditional Proof, ultra lean complete system of sentential logic, rules, the Rule of Disjunctive Proof

I should like to pose two questions in this paper: a. What becomes of the Rule of Conditional Proof on a reduction of the system of natural deduction in sentential logic to its bare essentials (an ultra lean complete system of sentential logic)? b. What rules and axioms are *required* in such a system?

The majority of textbooks we use in logic courses at both the introductory and intermediate levels employ some form of the method of natural deduction (proofs) and make use of the standard rules of inference and axioms of replacement.[1] These might include *Modus ponens*, Disjunctive Syllogism, [Disjunctive] Addition, DeMorgan's Theorems, Exportation and such like. Another standard in many of these books is the introduction of the Rule of Conditional Proof (hereafter RCP). This rule is said to "complete" the system. Tidman and Kahane write, for example, "Once we add this rule, our natural deduction procedure for sentential logic is complete, meaning that now every valid argument in sentential logic can be shown to be valid by means of a proof".[2] Tidman and Kahane are not alone in using RCP and noting that its inclusion "completes" the system. Others include Hurley, Layman and Herrick.

---

[1] A few of these books are: Hurley, Patrick J., *A Concise Introduction to Logic*, 8th edition (Wadsworth, 2003); Layman, C. Stephen, *The Power of Logic*, 3rd edition, McGraw/Hill, 2005; Tidman, Paul & Howard Kahane, *Logic & Philosophy: A Modern Introduction*, 8th edition, Wadsworth, 1999; Bergmann, Marrie, James Moor, & Jack Nelson, *The Logic Book*, 4th edition, McGraw/Hill, 2004; Copi, Irving M., *Symbolic Logic*, 5th edition, Macmillan, 1979; Barker, Stephen F., *The Elements of Logic*, 6th edition, McGraw/Hill, 2003; Michael F. Goodman, *First Logic*, 3rd edition, University Press of America.

[2] Tidman, *Ibid.*, p. 108.

Let us define RCP as follows: *From any assumption,* φ*, closure of the scope of the assumption is accomplished by deriving* ψ*, at which point* $\varphi \rightarrow \psi$ *is derived.* ψ is a formula derived either from the premises, and whatever assumption is open, or from other derived formulae in the proof.[3] A simple example of the use of RCP is as follows:[4]

1.	[(S & J) → C]	
2.	(J v ~S)	// (S → C)
3.	S	(Assumption)
4.	(~S v J)	2, Commutation
5.	J	3, 4 Disjunctive Syllogism
6.	(S & J)	3, 5 Conjunction
7.	C	1, 6 *Modus Ponens*
8.	(S → C)	3-7, RCP

Unlike RCP, there *are* rules that are unnecessary in the system. For instance, we can safely delete *Modus tollens* in favor of Material Implication and Disjunctive Syllogism. We can also omit one of the axioms of Association, namely

[p & (q & r)]<–> [(p & q) & r],

in favor of Simplification, Commutation and Conjunction.[5]

Not only may certain rules/axioms be deleted without altering the completeness of the system, if one dislikes the overpopulation of distinct symbols themselves, these may be reduced by three. W.V.O. Quine, for example, shows that such reductions are possible in *Mathematical Logic*.[6] We know that {~, v} is functionally complete, as is {~, &}.[7] Below is a list of common rules and axioms which have been “transformed” using only the tilde (negation) and the *vel* (disjunction). The common name is given for each rule/axiom.

Modus ponens: (~p v q) p // q

---

[3] Benson Mates’ use of Rule of Conditional Proof in his *Elementary Logic* is very tasty. He calls the rule *Conditionalization* and writes, “The sentence ***(N→P)*** may be entered on a line if ***P*** appears on an earlier line;...” (p. 112) and “In its principle use, rule C says that if you have succeeded in obtaining ***P*** from premises that include the sentence ***N***, then you may infer ***(N→P)*** and drop ***N*** from your list of premises.” (p. 115) Mates’ rule does not have the exact form of RCP because it does not specify that an assumption is made, but the essential aspect of deriving a conditional is the same.

[4] Key to symbols: &= for conjunction; →= for material conditional; v= for disjunction; ~= for negation; < **-** >= for logical equivalence, //= as the conclusion indicator. /// is used in the lean system for logical equivalence.

[5] One may also drop Disjunctive Syllogism in favor of Double Negation, Material Implication and *Modus ponens*. To keep *Modus tollens* out, if Disjunctive Syllogism were dropped, one could use Transposition and *Modus ponens*. Some of the accepted rules amount to nothing more than short-cuts for use within a proof procedure.

[6] Quine, W.V.O., *Mathematical Logic* (New York: Norton, 1940).

[7] A very interesting and succinct bit on functional completeness appears in Gerald Massey’s *Understanding Symbolic Logic*, New York: Harper & Row, 1970, ch. 13. Also of interest may be John Nolt’s *Logics*, Belmont, CA: Wadsworth, 1997. Whereas Massey uses truth tables as the vehicle for showing completeness, Nolt uses the truth tree method. If the trees are considered as founded on, and justified by, the tables, the reduction appears obvious.

	[Equiv to Disjunctive Syllogism]
Modus tollens:	(~p v q) ~q // ~p [Equiv to Disjunctive Syllogism]
Disjunctive Syllogism:	(p v q) ~p // q [Unchanged]
[Disjunctive] Addition:	p // (p v q) [Unchanged]
Conjunction:	p q // ~(~p v ~q)
Hypothetical Syllogism:	(~p v q) (~q v r) // (~p v r)
Simplification:	~(~p v ~q) // p [Redundant to Tautology b.]
Constructive Dilemma:	~[~(~p v q) v ~(~r v s)] (p v r) // (q v s)
DeMorgan's Theorems:	a. (~p v ~q) /// (~p v ~q) [Self-Redundant]
	b. ~(p v q) /// ~(p v q) [Self-Redundant]
Commutation:	a. (p v q) /// (q v p) [Unchanged]
	b. ~(~p v ~q) /// ~(~q v ~p) [Redundant to Commutation a.]
Association:	a. [p v (q v r)] /// [(p v q) v r] [Unchanged]
	b. ~[~p v (~q v ~r)] /// ~[(~p v ~q) v ~r] [Redundant to Association a.]
Distribution:	a. ~[~p v ~(q v r)] /// [~(~p v ~q) v ~(~p v ~r)]
	b. [p v ~(~q v ~r)] /// ~[~(p v q) v ~(p v r)]
Double Negation:	p /// ~ ~p [Unchanged]
Material Implication:	(~p v q) /// (~p v q) [Self-Redundant]

Transposition:	(~p v q) [Equiv to Commutation]	///	(q v ~p)
Exportation:	[(~p v ~q) v r]	///	[~p v (~q v r)]
Tautology:	a. p [Unchanged]	///	(p v p)
	b. p	///	~(~p v ~p)

Without further ado, let us ask what becomes of the Rule of Conditional Proof within this system? If we drop the conditional sentence in favor of the disjunction (with the first disjunct negated), there will be no place for line #8 in the proof above. It is not as though we can just leave line #8 as it is and use Material Implication to get to (~S v C),in line #8, because Material Implication will have been excluded as unnecessary (since there will be no conditionals to turn into, or replace, disjunctions). For all that, we would not desire to exclude RCP from the system, of course, for it is a *complete-making rule*. So, perhaps there is a way to save it in some form. One suggestion may be that we opt for a proof procedure that uses *reductio ad absurdum* (Indirect Proof, as it is called by some logicians). This is a nonstarter, however, for *reductio* proofs themselves rely on RCP. The reason I say this is that, while many logicians countenance going from

φ & ~φ

to

ψ,

a) we won't have conjunctions in the system, and b) this move hides the fact that knowing *that* everything follows from a contradiction is not the same as showing *how*, in fact, this is true (that is, how to show/prove it). However, showing that anything follows from a contradiction using a *reductio* proof requires assuming the negation of the conclusion of the argument, deriving a contradiction, deriving the conclusion itself on a line within the scope of the assumption, then closing the scope of the assumption, deriving a conditional. What one gets, then, is a conditional that is subsequently turned into a disjunction via Material Implication. A simple example of this, using a conditional (line #14) in the proof for convenience, is:

1.	(L v ~S)		
2.	(~L v D)	// (~S v D)	
3.	~(~S v D)		Assumption, Denial of Conclusion
4.	(~ ~S & ~D)		3, DeMorgan's
5.	(S & ~D)		4, Double Negation
6.	(D v ~L)		2, Commutation
7.	~D		5, Commutation, Simplification
8.	~L		6,7 Disjunctive Syllogism
9.	(~S v L)		1, Commutation
10.	S		5, Simplification
11.	L		9,10 Disjunctive Syllogism
12.	[L v (~S v D)]		11, Disjunctive Addition
13.	(~S v D)		8,12 Disjunctive Syllogism

14.	~(~S v D) → (~S v D)	3-13, Rule of Conditional Proof
15.	~ ~(~S v D) v (~S v D)	14, Material Implication
16.	(~S v D) v (~S v D)	15, Double Negation
17.	(~S v D)	16, Tautology

If one were to allow the following move, it would hide the conditional on line #14.

12a.	(L & ~L)	8,11 Conjunction
13a.	(~S v D)	3-12a *Reductio ad Absurdum*

However, just because the conditional does not appear in the proof does not mean that it is not there *tacitly*. What 12a and 13a constitute is a short-cut *Reductio*, or, what is perhaps a better way of saying it, using *Reductio ad absurdum* as a rule rather than a method.[8] Lines 12-17 use *Reductio* as a method.

It is also important that '12a' above is a conjunction and conjunctions, in the form of '*φ & ψ* ' will not be allowed in the system where the only logical symbols will be '~' and 'v'.

What I would like to propose is that since conditionals will not be allowed within the "lean" system (and conjunctions either, for that matter, though they are included above for convenience), and given that a rule doing the work of RCP is required for a complete system, we adopt the following rule, call it Rule of Disjunctive Proof. *From any assumption, **φ**, closure of the scope of the assumption is accomplished by deriving **ψ**, at which point **~φ v ψ** is derived.* A simple example is:

1.	(M v ~T)	
2.	~(~T v ~R)	// (M v ~R)
3.	R	Assumption
4.	T	2, Rule of Premise[9]
5.	M	1,4 Disjunctive Syllogism
6.	(~R v M)	3-5, Rule of Disjunctive Proof (RDP)
7.	(M v ~R)	6, Commutation.

If each rule and axiom that is either redundant or reductive to some other rule or axiom is deleted from the system, and if we include *Rule of Disjunctive Proof*, we end up with the following rules and axioms for a complete system of sentential logic:

Disjunctive Syllogism:	(p v q)		~p	//	q
Disjunctive Addition:	p	//	(p v q)		
Disjunctive Negation:		p	q	//	~(~p v ~q)

[8] I am not arguing that one should use *Reductio ad absurdum* as a method rather than a rule. Both are legitimate and can have their place within the proof procedure. Tidman & Kahane use the *Reductio* as a rule, as do Bergmann, Moore & Nelson, Hurley, Herrick, Layman and Copi (op cit.). Mates, on the other hand, seems to use it as a method (op cit. p. 119f).

[9] The concept of a "Rule of Premise" comes from the late Herbert Hendry who used this wording to indicate that the inference is valid without specifying the exact set of rules/axioms used in deriving the formula. In the present case, in the non-lean system, the rules would be DeMorgan's, Simplification and Double Negation. Mates calls this *Tautological inference*, op. cit. p. 112. The deduction in this proof from line 2 to line 4 would be accomplished by the following rules: DeMorgan's, Double Negation, Simplification.

Disjunctive Chain:	(~p v q) (~q v r) // (~p v r)
Constructive Dilemma:	~[~(~p v q) v ~(~r v s)] (p v r) // (q v s)
Commutation:	(p v q) /// (q v p)
Association:	[p v (q v r)] /// [(p v q) v r]
Distribution:	a. ~[~p v ~(q v r)] /// [~(~p v ~q) v ~(~p v ~r)]
	b. [p v ~(~q v ~r)] /// ~[~(p v q) v ~(p v r)]
Double Negation:	p /// ~ ~p
Tautology:	a. p /// (p v p)
	b. p /// ~(~p v ~p)
Rule of Disjunctive Proof:	Assume p, derive q, infer (~p v q)

Use of the individual rules/axioms becomes an exercise in specification, almost busy-work. However, if one admits the Rule of Premise (or Mates' Conditionalization rule), specifying any rules whatever seems almost arbitrary. That is, since Rule of Premise would take the place of a Disjunctive Addition move, or a Disjunctive Chain move, or a Commutation, it would also take the place of the use of these three at one time, as so: from "(Li v ~Ji)" and "(~Li v Ki)", we could derive "[(Ki v ~Ji) v Ji]".

As a teaching tool, it could be argued that the set of rules under {~,v} would be very unintuitive as compared with the rules under {~, v, → , < - > , &}, not to mention downright tedious. As a system, {~,v} seems far superior to {~, v, → , < - > , &} for its simplicity as well as elegance.